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Some Theorems Associated with Finite Difference Calculus and its Applications in Hypergeometric Reduction Formulae

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ABSTRACT: By means of finite difference calculus and series iteration technique, we give alternative proofs of Montmort's theorem and S. Narayana Aiyar theorem found in Chrystal's Algebra. In this paper we obtain some hypergeometric reduction formulae for $_3F_4$, $_4F_5$, $_5F_6$, $_6F_7$ whose numerator and denominator parameters are complex numbers.

Keywords: Finite difference Calculus, Gamma function, Generalized hypergeometric function of one variable, Montmort's theorem, Pochhammer symbol, Reduction formula, S. Narayana Aiyar theorem.

I. INTRODUCTION

The Pochhammer symbol or generalized factorial function or shifted factorial is defined by

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{bmatrix} 1; n=0 \\ a(a+1)(a+2)...(a+n-1); n=1,2,3... \end{bmatrix} \tag{1}$$

where $a\neq 0,-1,-2,...$ and the notation 'Γ' stands for Gamma function.

$$(a)_{-n}=\frac{\Gamma(a-n)}{\Gamma(a)}=\frac{(-1)^n}{(1-a)_n} \tag{2}$$

where $a\neq ...-3,-2,-1,0,1,2,3,...$ and $n=1,2,3,...$

If $m=1,2,3,...$ and $n=0,1,2,3,...$ then

$$(b)_{mn}=m^{mn}\left(\frac{b}{m}\right)_n\left(\frac{b+1}{m}\right)_n...\left(\frac{b+m-2}{m}\right)_n\left(\frac{b+m-1}{m}\right)_n \tag{3}$$

The notation denotes $\Delta(N;b)$ the array of N parameters given by $\frac{b}{N},\frac{b+1}{N},...,\frac{b+n-1}{N}$. where $N=1,2,3,...$

$$(\alpha)_{p+q}=(\alpha)_p(\alpha+p)_q=(\alpha)_q(\alpha+q)_p \tag{4}$$

$$\left[(a_A)\right]_n=(a_1)_n(a_2)_n...(a_A)_n=\prod_{m=1}^{A}(a_m)_n=\prod_{m=1}^{A}\frac{\Gamma(a_m+n)}{\Gamma(a_m)} \tag{5}$$

where $a_1,a_2,...,a_A,b_1,b_2,...,b_B$ and z may be real and complex numbers.
The generalized hypergeometric function of one variable [4,5] is defined as

$${}_AF_B\left[\begin{matrix}a_1,a_2,...,a_A;\\ b_1,b_2,...,b_B;\end{matrix}z\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n...(a_A)_n}{(b_1)_n(b_2)_n...(b_B)_n}\frac{z^n}{n!} \quad\text{or}\quad {}_AF_B\left[\begin{matrix}(a_A);\\ (b_B);\end{matrix}z\right]=\sum_{n=0}^{\infty}\frac{[(a_A)]_n}{[(b_B)]_n}\frac{z^n}{n!} \tag{6}$$

Where for the sake of convenience (in the contracted notation), (a_A) denotes the array of '*A*' number of parameters given by a_1, a_2, …. a_A. The denominator parameters are neither zero nor negative integers. The numerator parameters may be zero and negative integers. A and B are positive integers or zero. Empty sum is to be interpreted as zero and empty product as unity. $\sum_{n=a}^{b}$ and $\prod_{n=a}^{b}$ are empty if $b<a$.

Suppose that the numerator parameters are neither zero nor negative integers (otherwise the question of convergence will not arise).
If $A=B+1$, the series $_AF_B$ converges for $|z|<1$ and diverges for $|z|>1$
Lemma: If a, p and n are suitably adjusted real or complex numbers such that associated pochhammer's symbols are well defined, then we have

$$(a+pn)=\frac{a\left(\frac{a+p}{p}\right)_n}{\left(\frac{a}{p}\right)_n} \tag{7}$$

Proof: Using the definition of pochhammer's symbol $(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$ and recurrence relation $\Gamma(z+1)=z\,\Gamma(z)$, then we have

$$\frac{a\left(\frac{a+p}{p}\right)_n}{\left(\frac{a}{p}\right)_n}=\frac{a\,\Gamma\left(\frac{a+p}{p}+n\right)\,\Gamma\left(\frac{a}{p}\right)}{\Gamma\left(\frac{a+p}{p}\right)\Gamma\left(\frac{a}{p}+n\right)}$$

$$=\frac{a\,\Gamma\left(1+\frac{a}{p}+n\right)\,\Gamma\left(\frac{a}{p}\right)}{\Gamma\left(1+\frac{a}{p}\right)\Gamma\left(\frac{a}{p}+n\right)}$$

$$=\frac{a\,\left(\frac{a}{p}+n\right)\Gamma\left(\frac{a}{p}+n\right)\,\Gamma\left(\frac{a}{p}\right)}{\left(\frac{a}{p}\right)\Gamma\left(\frac{a}{p}\right)\Gamma\left(\frac{a}{p}+n\right)}$$

$$=(a+pn)$$

II. MAIN REDUCTION FORMULA

$$ {}_4F_5\left[\begin{matrix}\frac{2\sqrt{2}-1-i}{2\sqrt{2}},\frac{2\sqrt{2}-1+i}{2\sqrt{2}},\frac{2\sqrt{2}+1-i}{2\sqrt{2}},\frac{2\sqrt{2}+1+i}{2\sqrt{2}};\\ \frac{1}{2},\frac{-1-i}{2\sqrt{2}},\frac{-1+i}{2\sqrt{2}},\frac{1-i}{2\sqrt{2}},\frac{1+i}{2\sqrt{2}};\end{matrix}\ \frac{x^2}{4}\right]=(1+7x^2+x^4)\cosh(x)+(x+6x^3)\sinh(x) \tag{8}$$

$$ {}_4F_5\left[\begin{matrix}\frac{3\sqrt{2}-1-i}{2\sqrt{2}},\frac{3\sqrt{2}-1+i}{2\sqrt{2}},\frac{3\sqrt{2}+1-i}{2\sqrt{2}},\frac{3\sqrt{2}+1+i}{2\sqrt{2}};\\ \frac{3}{2},\frac{\sqrt{2}-1-i}{2\sqrt{2}},\frac{\sqrt{2}-1+i}{2\sqrt{2}},\frac{\sqrt{2}+1-i}{2\sqrt{2}},\frac{\sqrt{2}+1+i}{2\sqrt{2}};\end{matrix}\ \frac{x^2}{4}\right]=\left(\frac{1}{2}x^3+\frac{7}{2}x+\frac{1}{2x}\right)\sinh(x)+\left(3x^2+\frac{1}{2}\right)\cosh(x) \tag{9}$$

$$ {}_5F_6\left[\begin{matrix}\frac{3}{2},\frac{7-\sqrt{5}-i\sqrt{10-2\sqrt{5}}}{8},\frac{7-\sqrt{5}+i\sqrt{10-2\sqrt{5}}}{8},\frac{7+\sqrt{5}-i\sqrt{10+2\sqrt{5}}}{8},\frac{7+\sqrt{5}+i\sqrt{10+2\sqrt{5}}}{8};\\ 1,\frac{1}{2},\frac{-1-\sqrt{5}-i\sqrt{10-2\sqrt{5}}}{8},\frac{-1-\sqrt{5}+i\sqrt{10-2\sqrt{5}}}{8},\frac{-1+\sqrt{5}-i\sqrt{10+2\sqrt{5}}}{8},\frac{-1+\sqrt{5}+i\sqrt{10+2\sqrt{5}}}{8};\end{matrix}\ \frac{x^2}{4}\right]$$
$$=(1+15x^2+10x^4)\cosh(x)+(x+25x^3+x^5)\sinh(x) \tag{10}$$

$$ {}_5F_6\left[\begin{matrix}2,\frac{11-\sqrt{5}-i\sqrt{10-2\sqrt{5}}}{8},\frac{11-\sqrt{5}+i\sqrt{10-2\sqrt{5}}}{8},\frac{11+\sqrt{5}-i\sqrt{10+2\sqrt{5}}}{8},\frac{11+\sqrt{5}+i\sqrt{10+2\sqrt{5}}}{8};\\ 1,\frac{3}{2},\frac{3-\sqrt{5}-i\sqrt{10-2\sqrt{5}}}{8},\frac{3-\sqrt{5}+i\sqrt{10-2\sqrt{5}}}{8},\frac{3+\sqrt{5}-i\sqrt{10+2\sqrt{5}}}{8},\frac{3+\sqrt{5}+i\sqrt{10+2\sqrt{5}}}{8};\end{matrix}\ \frac{x^2}{4}\right]$$
$$=\left(5x^3+\frac{15}{2}x+\frac{1}{2x}\right)\sinh(x)+\left(\frac{1}{2}x^4+\frac{25}{2}x^2+\frac{1}{2}\right)\cosh(x) \tag{11}$$

$$ {}_3F_4\left[\begin{matrix}\frac{7}{3},\frac{-i\sqrt{7}}{2},\frac{i\sqrt{7}}{2};\\ \frac{1}{2},\frac{4}{3},\frac{-2-i\sqrt{7}}{2},\frac{-2+i\sqrt{7}}{2};\end{matrix}\ \frac{x^2}{4}\right]=\left(1+\frac{5}{88}x^2\right)\cosh(x)+\left(\frac{3}{88}x^3\right)\sinh(x) \tag{12}$$

$$ {}_3F_4\left[\begin{matrix}\frac{17}{6},\frac{1-i\sqrt{7}}{2},\frac{1+i\sqrt{7}}{2};\\ \frac{11}{6},\frac{3}{2},\frac{-1-i\sqrt{7}}{2},\frac{-1+i\sqrt{7}}{2};\end{matrix}\ \frac{x^2}{4}\right]=\left(\frac{1}{x}+\frac{5}{88}x\right)\sinh(x)+\left(\frac{3}{88}x^2\right)\cosh(x) \tag{13}$$

$$ {}_3F_4\left[\begin{matrix}\frac{1}{2}-\cos\frac{2\pi}{9},\frac{1}{2}-\cos\frac{4\pi}{9},\frac{1}{2}-\cos\frac{8\pi}{9};\\ \frac{1}{2},-\frac{1}{2}-\cos\frac{2\pi}{9},-\frac{1}{2}-\cos\frac{4\pi}{9},-\frac{1}{2}-\cos\frac{8\pi}{9};\end{matrix}\ \frac{x^2}{4}\right]=\cosh(x)+\left(\frac{1}{3}x^3-\frac{2}{3}x\right)\sinh(x) \tag{14}$$

$$ {}_3F_4\left[\begin{matrix}1-\cos\frac{2\pi}{9},1-\cos\frac{4\pi}{9},1-\cos\frac{8\pi}{9};\\ \frac{3}{2},-\cos\frac{2\pi}{9},-\cos\frac{4\pi}{9},-\cos\frac{8\pi}{9};\end{matrix}\ \frac{x^2}{4}\right]=\left(\frac{3}{x}\right)\sinh(x)+(x^2-2)\cosh(x) \tag{15}$$

$$ {}_4F_5\left[\begin{matrix}\frac{3-i}{4},\frac{3+i}{4},\frac{4-\sqrt{2}}{2},\frac{4+\sqrt{2}}{2};\\ \frac{1}{2},\frac{-1-i}{2},\frac{-1+i}{2},\frac{2-\sqrt{2}}{2},\frac{2+\sqrt{2}}{2};\end{matrix}\ \frac{x^2}{4}\right]=\left(x^4+\frac{29}{2}x^2+1\right)\cosh(x)+\left(\frac{5}{2}x+9x^3\right)\sinh(x) \tag{16}$$

$$ {}_4F_5\left[\begin{matrix}\frac{5-i}{4},\frac{5+i}{4},\frac{5-\sqrt{2}}{2},\frac{5+\sqrt{2}}{2}; \\ \frac{3}{2},\frac{1-i}{4},\frac{1+i}{4},\frac{3-\sqrt{2}}{2},\frac{3+\sqrt{2}}{2};\end{matrix}\ \frac{x^2}{4}\right] = \left(\frac{2}{5x}+\frac{29}{5}x+\frac{2}{5}x^3\right)\sinh(x)+\left(1+\frac{18}{5}x^2\right)\cosh(x) \tag{17} $$

$$ {}_6F_7\left[\begin{matrix}\frac{4\sqrt{2}-3(\sqrt{3}+1)+3(\sqrt{3}-1)i}{4\sqrt{2}},\frac{4\sqrt{2}+3(\sqrt{3}-1)-3(\sqrt{3}+1)i}{4\sqrt{2}},\frac{4\sqrt{2}+3(\sqrt{3}+1)-3(\sqrt{3}-1)i}{4\sqrt{2}}, \\ \frac{1}{2},\frac{-3(\sqrt{3}+1)+3(\sqrt{3}-1)i}{4\sqrt{2}},\frac{3(\sqrt{3}-1)-3(\sqrt{3}+1)i}{4\sqrt{2}},\frac{3(\sqrt{3}+1)-3(\sqrt{3}-1)i}{4\sqrt{2}},\end{matrix}\right. $$

$$ \left.\begin{matrix}\frac{4\sqrt{2}-3(\sqrt{3}-1)+3(\sqrt{3}+1)i}{4\sqrt{2}},\frac{2\sqrt{2}-3-3i}{2\sqrt{2}},\frac{2\sqrt{2}+3-3i}{2\sqrt{2}}; \\ \frac{-3(\sqrt{3}-1)+3(\sqrt{3}+1)i}{4\sqrt{2}},\frac{3-3i}{2\sqrt{2}},\frac{-3-3i}{2\sqrt{2}};\end{matrix}\ \frac{x^2}{4}\right] $$

$$ =\frac{1}{729}[(729i+31x^2+65x^4+x^6)\cosh(x)+(x+9x^3+15x^5)\sinh(x)] \tag{18} $$

$$ {}_6F_7\left[\begin{matrix}\frac{6\sqrt{2}-3(\sqrt{3}+1)+3(\sqrt{3}-1)i}{4\sqrt{2}},\frac{6\sqrt{2}+3(\sqrt{3}-1)-3(\sqrt{3}+1)i}{4\sqrt{2}},\frac{6\sqrt{2}+3(\sqrt{3}+1)-3(\sqrt{3}-1)i}{4\sqrt{2}}, \\ \frac{3}{2},\frac{2\sqrt{2}-3(\sqrt{3}+1)+3(\sqrt{3}-1)i}{4\sqrt{2}},\frac{2\sqrt{2}+3(\sqrt{3}+1)-3(\sqrt{3}-1)i}{4\sqrt{2}},\frac{2\sqrt{2}-3(\sqrt{3}-1)+3(\sqrt{3}+1)i}{4\sqrt{2}},\end{matrix}\right. $$

$$ \left.\begin{matrix}\frac{6\sqrt{2}-3(\sqrt{3}-1)+3(\sqrt{3}+1)i}{4\sqrt{2}},\frac{3\sqrt{2}-3-3i}{2\sqrt{2}},\frac{3\sqrt{2}+3-3i}{2\sqrt{2}}; \\ \frac{2\sqrt{2}+3(\sqrt{3}-1)-3(\sqrt{3}+1)i}{4\sqrt{2}},\frac{\sqrt{2}-3-3i}{2\sqrt{2}},\frac{\sqrt{2}+3-3i}{2\sqrt{2}};\end{matrix}\ \frac{x^2}{4}\right] $$

$$ =\left\{\frac{(160+27\sqrt{2})+(72-240\sqrt{2})i}{147442-8640\sqrt{2}}\right\}\times\left[\left(\frac{729i}{x}+31x+54x^3+x^5\right)\sinh(x)+(1+90x^2+15x^4)\cosh(x)\right] \tag{19} $$

III. SOME THEOREMS ASSOCIATED WITH FINITE DIFFERENCE CALCULUS

Suppose $\{\emptyset(n)\}_{n=0}^{\infty}$ is the sequence then

$$ \emptyset(n)=\sum_{k=0}^{n}\binom{n}{k}\Delta^k\emptyset(0);\ \Delta^0\emptyset(n)=\emptyset(n)\ \ and\ \ \Delta^0\emptyset(0)=\emptyset(0) \tag{20} $$

where Δ^k denotes the k^{th} order ordinary difference in forward notation and n is a non-negative integer (i.e. n=0,1,2,3,…) and $\binom{n}{k}$ is Binomial coefficients $=\frac{n!}{k!(n-k)!}$.

Proof: We shall prove above identity by means of mathematical induction.

Put $n = 1$ in Eqn. (20), we get

$\Delta\emptyset(0)=\emptyset(1)-\emptyset(0)$

which is true by the basic definition of forward differences.

Suppose it is true for a particular integral value of $n(=p)$, then

$$ \emptyset(p)=\sum_{k=0}^{p}\binom{p}{k}\Delta^k\emptyset(0) $$

If the interval of differencing in the consecutive values of p, is unity then

$\Delta\emptyset(p)=\emptyset(p+1)-\emptyset(p)=\nabla\emptyset(p+1)$

where ∇ is the backward ordinary difference operator.

Therefore,

$$ \emptyset(p+1)=\Delta\emptyset(p)+\emptyset(p) $$

$$ \emptyset(p+1)=\sum_{k=0}^{p}\binom{p}{k}\Delta^k\emptyset(0)\ +\Delta\left[\sum_{k=0}^{p}\binom{p}{k}\Delta^k\emptyset(0)\right] $$

$$ =\emptyset(0)+\sum_{k=1}^{p}\binom{p}{k}\Delta^k\emptyset(0)\ +\sum_{k=0}^{p}\binom{p}{k}\Delta^{k+1}\emptyset(0) $$

$$ =\emptyset(0)+\sum_{k=1}^{p}\binom{p}{k}\Delta^k\emptyset(0)\ +\sum_{k=0}^{p-1}\binom{p}{k}\Delta^{k+1}\emptyset(0)+\Delta^{p+1}\emptyset(0) $$

$$ =\emptyset(0)+\sum_{k=1}^{p}\binom{p}{k}\Delta^k\emptyset(0)\ +\sum_{k=1}^{p}\binom{p}{k-1}\Delta^k\emptyset(0)+\Delta^{p+1}\emptyset(0) $$

$$= \emptyset(0) + \sum_{k=1}^{p} \Delta^k \emptyset(0) \left\{\binom{p}{k} + \binom{p}{k-1}\right\} + \Delta^{p+1}\emptyset(0)$$

$$= \emptyset(0) + \sum_{k=1}^{p} \Delta^k \emptyset(0) \left\{ {p+1 \atop k} \right\} + \Delta^{p+1}\emptyset(0)$$

$$= \sum_{k=0}^{p} \Delta^k \emptyset(0) \left\{ {p+1 \atop k} \right\} + \Delta^{p+1}\emptyset(0)$$

$$\emptyset(p+1) = \textstyle\sum_{k=0}^{p+1} \Delta^k \emptyset(0) \left\{ {p+1 \atop k} \right\} \tag{21}$$

It means Eqn. (20) is true for any positive integral value of *n*.
The following general theorems may be of use to those who take some special interest in summation of series.

$$\Delta^k \emptyset(n) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \emptyset(n+j);\;\; \Delta^0\emptyset(n) = \emptyset(n) \tag{22}$$

where $\Delta^k\phi(n)$ denotes the n^{th} term of k^{th} order ordinary forward difference derived from the sequence $\{\emptyset(n)\}_{n=0}^{\infty}$.
Put $k = 1$

$$\Delta\emptyset(n) = \emptyset(n+1) - \emptyset(n)$$

which is true by the basic definition of forward difference, when interval of differencing is unity. Let it be true for all positive integral values of *k* (i.e. *m*)

$$\Delta^m \emptyset(n) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \emptyset(n+j)$$

Now we shall prove for next value of *k* (i.e. *m*+1)

$$\Delta^{m+1} \emptyset(n) = \sum_{j=0}^{m+1} (-1)^{m+1-j} \binom{m+1}{j} \emptyset(n+j)$$

Proof:

$$\Delta^{m+1}\emptyset(n) = \Delta[\Delta^m\emptyset(n)]$$

$$= \Delta[\Delta^{m-1}\emptyset(n+1) - \Delta^{m-1}\emptyset(n)]$$

$$= \Delta^m\emptyset(n+1) - \Delta^m\emptyset(n)$$

$$= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \emptyset(n+j+1) \; - \; \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \emptyset(n+j)$$

$$= \sum_{j=0}^{m-1} (-1)^{m-j} \binom{m}{j} \emptyset(n+j+1) + (-1)^0 \binom{m}{m} \emptyset(n+m+1) \; - \; \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \emptyset(n+j)$$

$$= \sum_{j=1}^{m} (-1)^{m-j+1} \binom{m}{j-1} \emptyset(n+j) + (-1)^0 \binom{m}{m} \emptyset(n+m+1) \; - \; \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \emptyset(n+j)$$

$$= \sum_{j=1}^{m} (-1)^{m-j+1} \binom{m}{j-1} \emptyset(n+j) + \emptyset(n+m+1) - (-1)^m\emptyset(n) - \sum_{j=1}^{m} (-1)^{m-j} \binom{m}{j} \emptyset(n+j)$$

$$= \emptyset(n+m+1) - (-1)^m\emptyset(n) - \sum_{j=1}^{m} (-1)^{m-j} \binom{m}{j-1} \emptyset(n+j) \; - \; \sum_{j=1}^{m} (-1)^{m-j} \binom{m}{j} \emptyset(n+j)$$

$$= \emptyset(n+m+1) - (-1)^m\emptyset(n) - \sum_{j=1}^{m} (-1)^{m-j} \left\{\binom{m}{j-1} + \binom{m}{j}\right\} \emptyset(n+j)$$

Now using Binomial identity $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$, we get

$$\Delta^{m+1}\emptyset(n) = \emptyset(n+m+1) + (-1)^{m+1}\emptyset(n) + \sum_{j=1}^{m} (-1)^{m-j+1} \binom{m+1}{j} \emptyset(n+j)$$

$$\Delta^{m+1}\emptyset(n) = \emptyset(n+m+1) + \sum_{j=0}^{m} (-1)^{m-j+1} \binom{m+1}{j} \emptyset(n+j)$$

$$\Delta^{m+1}\emptyset(n) = \sum_{j=0}^{m+1} (-1)^{m-j+1} \binom{m+1}{j} \emptyset(n+j) \tag{23}$$

Therefore

$$\Delta^k \emptyset(n) = \sum_{j=0}^{k} \ (-1)^{k-j} \binom{k}{j} \emptyset(n+j)$$

is true for all positive integer value of *k*.

Montmort's Theorem: This theorem [2, 3] state that when ϕ(*n*) ≠ ±∞ at n=0,1,2,3,…. then

$$\sum_{n=0}^{\infty} \emptyset(n)x^n = \left(\frac{1}{1-x}\right)\sum_{k=0}^{\infty} \Delta^k\emptyset(0) \left(\frac{x}{1-x}\right)^k \tag{24}$$

The proof of above theorem can be found in Chrystal's Algebra [3]. Here we give another proof of Montmort's theorem using series manipulation technique.

Proof:

$$\sum_{n=0}^{\infty} \emptyset(n)x^n = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{n}{k} \Delta^k\emptyset(0)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} x^{n+k} \binom{n+k}{k} \Delta^k\emptyset(0) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} x^{n+k} \frac{(n+k)!}{k!\,n!}\Delta^k\emptyset(0)$$

$$= \sum_{k=0}^{\infty} \frac{(1)_k \Delta^k\emptyset(0)\, x^k}{k!}\sum_{n=0}^{\infty}\frac{(1+k)_n \ x^n}{n!}$$

$$= \sum_{k=0}^{\infty} \Delta^k\emptyset(0)\, x^k\, (1-x)^{-1-k}$$

$$= \left(\frac{1}{1-x}\right)\sum_{k=0}^{\infty} \Delta^k\emptyset(0) \left(\frac{x}{1-x}\right)^k = \left(\frac{1}{1-x}\right)\sum_{n=0}^{\infty} \Delta^n\emptyset(0) \left(\frac{x}{1-x}\right)^n$$

The right hand side is rapidly convergent when ϕ(*n*) is a rational integral algebraic function of *n*.

S. Narayana Aiyar (Iyar or Ayyar) Theorem [1]: Ramanujan's closest mathematical friend in India, Chief accountant at the Madras port trust office [2]. When ϕ(*n*)≠ ±∞ at n=0,1,2,3,…. then

$$\sum_{n=0}^{\infty} \emptyset(n)\frac{x^n}{n!} = e^x\sum_{n=0}^{\infty} \Delta^n\emptyset(0)\ \frac{x^n}{n!} \tag{25}$$

Proof:

$$\sum_{n=0}^{\infty} \emptyset(n)\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{k=0}^{n} \binom{n}{k} \Delta^k\emptyset(0)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^{n+k}}{(n+k)!}\binom{n+k}{k} \Delta^k\emptyset(0) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^{n+k}}{(n+k)!}\,\frac{(n+k)!}{k!\,n!}\Delta^k\emptyset(0)$$

$$= \sum_{k=0}^{\infty} \Delta^k\emptyset(0)\ \frac{x^k}{k!}\ \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty} \emptyset(n)\frac{x^n}{n!} = e^x\sum_{k=0}^{\infty} \Delta^k\emptyset(0)\ \frac{x^k}{k!}$$

Since summation index *k* is dummy variable therefore we have,

$$\sum_{n=0}^{\infty} \emptyset(n)\frac{x^n}{n!} = e^x\sum_{n=0}^{\infty} \Delta^n\emptyset(0)\ \frac{x^n}{n!}$$

The right hand side is rapidly convergent when ϕ(*n*) is a rational integral algebraic function of *n*. Similar theorems may be derived by changing the sign of *x* on both sides or by integrating both the sides between fixed limits.

Now applying the series identity

$$\sum_{n=0}^{\infty} A(n) = \sum_{n=0}^{\infty} A(2n) + \sum_{n=0}^{\infty} A(2n+1)$$

In Eqns. (25), we get

$$\sum_{n=0}^{\infty} \emptyset(2n)\frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \emptyset(2n+1)\frac{x^{2n+1}}{(2n+1)!} = e^x\sum_{n=0}^{\infty} \Delta^{2n}\emptyset(0)\ \frac{x^{2n}}{(2n)!} + e^x\sum_{n=0}^{\infty} \Delta^{2n+1}\emptyset(0)\ \frac{x^{2n+1}}{(2n+1)!} \tag{26}$$

Replacing *x* by *ix* and equating the real and imaginary parts, we get

$$\sum_{n=0}^{\infty}(-1)^n\emptyset(2n)\frac{x^{2n}}{(2n)!}=\cos x\sum_{n=0}^{\infty}(-1)^n\,\Delta^{2n}\emptyset(0)\,\frac{x^{2n}}{(2n)!}-\sin x\sum_{n=0}^{\infty}(-1)^n\Delta^{2n+1}\emptyset(0)\,\frac{x^{2n+1}}{(2n+1)!} \quad (27)$$

$$\sum_{n=0}^{\infty}(-1)^n\emptyset(2n+1)\frac{x^{2n+1}}{(2n+1)!}=\sin x\sum_{n=0}^{\infty}(-1)^n\,\Delta^{2n}\emptyset(0)\,\frac{x^{2n}}{(2n)!}+\cos x\sum_{n=0}^{\infty}(-1)^n\,\Delta^{2n+1}\emptyset(0)\,\frac{x^{2n+1}}{(2n+1)!} \quad (28)$$

Now again replacing *x* by *ix* in equation (25) and (28), we get

$$\sum_{n=0}^{\infty}\emptyset(2n)\frac{x^{2n}}{(2n)!}=\cosh(x)\sum_{n=0}^{\infty}\Delta^{2n}\emptyset(0)\,\frac{x^{2n}}{(2n)!}+\sinh(x)\sum_{n=0}^{\infty}\Delta^{2n+1}\emptyset(0)\,\frac{x^{2n+1}}{(2n+1)!}$$

$$\sum_{n=0}^{\infty}\emptyset(2n)\frac{x^{2n}}{(2n)!}=\cosh(x)\left[\emptyset(0)+\Delta^2\emptyset(0)\,\frac{x^2}{2!}+\Delta^4\emptyset(0)\,\frac{x^4}{4!}+\Delta^6\emptyset(0)\,\frac{x^6}{6!}+\cdots\right]$$
$$+\sinh(x)\left[\Delta\emptyset(0)\,\frac{x}{1!}+\Delta^3\emptyset(0)\,\frac{x^3}{3!}+\Delta^5\emptyset(0)\,\frac{x^5}{5!}+\Delta^7\emptyset(0)\,\frac{x^7}{7!}+\cdots\right] \quad (29)$$

$$\sum_{n=0}^{\infty}\emptyset(2n+1)\frac{x^{2n+1}}{(2n+1)!}=\sinh(x)\sum_{n=0}^{\infty}\Delta^{2n}\emptyset(0)\,\frac{x^{2n}}{(2n)!}+\cosh(x)\sum_{n=0}^{\infty}\Delta^{2n+1}\emptyset(0)\,\frac{x^{2n+1}}{(2n+1)!}$$

$$\sum_{n=0}^{\infty}\emptyset(2n+1)\frac{x^{2n+1}}{(2n+1)!}$$
$$=\sinh(x)\left[\emptyset(0)+\Delta^2\emptyset(0)\,\frac{x^2}{2!}+\Delta^4\emptyset(0)\,\frac{x^4}{4!}+\Delta^6\emptyset(0)\,\frac{x^6}{6!}+\cdots\right]$$
$$+\cosh(x)\left[\Delta\emptyset(0)\,\frac{x}{1!}+\Delta^3\emptyset(0)\,\frac{x^3}{3!}+\Delta^5\emptyset(0)\,\frac{x^5}{5!}+\Delta^7\emptyset(0)\,\frac{x^7}{7!}+\cdots\right] \quad (30)$$

IV. SOME TYPICAL LINEAR FACTORS

Suppose $\phi(n)=n^4+1$ then using De-Movier's theorem

$$\emptyset(n)=\left[n+\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right]\left[n+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right]\left[n-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right]\left[n-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right] \quad (31)$$

Using De-Movier's theorem, we can factorise the polynomial $\phi(n)=n^5+1$ then

$$\emptyset(n)=(n+1)\left[n-\left(\frac{\sqrt{5}+1}{4}\right)-i\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right)\right]\left[n+\left(\frac{\sqrt{5}-1}{4}\right)-i\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)\right]$$
$$\times\left[n-\left(\frac{\sqrt{5}+1}{4}\right)+i\left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right)\right]\left[n+\left(\frac{\sqrt{5}+1}{4}\right)+i\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)\right] \quad (32)$$

Using Cardan's method, we can factorize the cubic polynomials $\phi(n)=3n^3-4n^2+n+88$ then

$$\emptyset(n)=\left[n-2-i\sqrt{7}\right]\left[n-2+i\sqrt{7}\right](3n+8) \quad (33)$$

If $\phi(n)=n^3-3n^2+3$ then

$$\emptyset(n)=\left[n-1-2\cos\frac{2\pi}{9}\right]\left[n-1-2\cos\frac{4\pi}{9}\right]\left[n-1-2\cos\frac{8\pi}{9}\right] \quad (34)$$

Using Ferrari's method or Descarte's method we can factories the biquadratic polynomial $\phi(n)=2n^4+6n^3-3n^2+2$ then

$$\emptyset(n)=[2n-1-i]\left[n-\frac{1}{2}+\frac{i}{2}\right][n+2-\sqrt{2}][n+2+\sqrt{2}] \quad (35)$$

If $\phi(n)=n^6+729i$ then

$$\emptyset(n)=\left[n-\frac{3(\sqrt{3}+1)}{2\sqrt{2}}+i\,\frac{3(\sqrt{3}-1)}{2\sqrt{2}}\right]\left[n+\frac{3(\sqrt{3}-1)}{2\sqrt{2}}-i\,\frac{3(\sqrt{3}+1)}{2\sqrt{2}}\right]$$
$$\times\left[n+\frac{3(\sqrt{3}+1)}{2\sqrt{2}}-i\,\frac{3(\sqrt{3}-1)}{2\sqrt{2}}\right]\left[n-\frac{3(\sqrt{3}-1)}{2\sqrt{2}}+i\,\frac{3(\sqrt{3}+1)}{2\sqrt{2}}\right]\times\left[n-\frac{3}{\sqrt{2}}-\frac{3i}{\sqrt{2}}\right]\left[n+\frac{3}{\sqrt{2}}-\frac{3i}{\sqrt{2}}\right] \quad (36)$$

V. SOME ORDINARY FORWARD DIFFERENCES

If points are equally spaced then n^{th} order ordinary differences for n^{th} degree polynomial function, will be constant and higher order ordinary differences will be zero.

Suppose $\phi(n)=n^4+1$ and interval of differencing in the consecutive values of *n* is unity then

$\Delta\emptyset(n)=\emptyset(n+1)-\emptyset(n)=4n^3+6n^2+4n+1$

$\Delta^2\emptyset(n)=12n^2+24n+14$

$\Delta^3 \emptyset(n) = 24n + 36$
$\Delta^4 \emptyset(n) = 24$
$\Delta^5 \emptyset(n) = \Delta^6 \emptyset(n) = \cdots = 0$
Therefore
$\emptyset(0) = 1, \Delta\, \emptyset(0) = 1, \Delta^2 \emptyset(0) = 14, \Delta^3 \emptyset(0) = 36, \Delta^4 \emptyset(0) = 24, \Delta^5 \emptyset(0) = \Delta^6 \emptyset(0) = \cdots = 0$
Now consider $\phi(n)=n^5+1$ and interval of differencing in the consecutive values of n is unity then
$\Delta \emptyset(n) = \emptyset(n+1) - \emptyset(n) = 5n^4 + 10n^3 + 10n^2 + 5n + 1$
$\Delta^2 \emptyset(n) = 20n^3 + 60n^2 + 70n + 30$
$\Delta^3 \emptyset(n) = 60n^2 + 180n + 150$
$\Delta^4 \emptyset(n) = 120n + 240$
$\Delta^5 \emptyset(n) = 120$
$\Delta^6 \emptyset(n) = \Delta^7 \emptyset(n) = \cdots = 0$
Therefore
$\emptyset(0) = 1, \Delta\, \emptyset(0) = 1, \Delta^2 \emptyset(0) = 30, \Delta^3 \emptyset(0) = 150, \Delta^4 \emptyset(0) = 240, \Delta^5 \emptyset(0) = 120,$
$\Delta^6 \emptyset(0) = \Delta^7 \emptyset(0) = \cdots = 0$
Now consider $\phi(n)=3n^3-4n^2+n+88$ and interval of differencing in the consecutive values of n is unity then
$\Delta \emptyset(n) = \emptyset(n+1) - \emptyset(n) = 9n^2 + n$
$\Delta^2 \emptyset(n) = 18n + 10$
$\Delta^3 \emptyset(n) = 18$
$\Delta^4 \emptyset(n) = \Delta^5 \emptyset(n) = \cdots = 0$
Therefore
$\emptyset(0) = 88, \Delta\, \emptyset(0) = 0, \Delta^2 \emptyset(0) = 10, \Delta^3 \emptyset(0) = 18, \Delta^4 \emptyset(0) = \Delta^5 \emptyset(0) = \cdots = 0$
Now consider $\phi(n)=n^3-3n^2+3$ and interval of differencing in the consecutive values of n is unity then
$\Delta \emptyset(n) = \emptyset(n+1) - \emptyset(n) = 3n^2 - 3n - 2$
$\Delta^2 \emptyset(n) = 6n$
$\Delta^3 \emptyset(n) = 6$
$\Delta^4 \emptyset(n) = \Delta^5 \emptyset(n) = \cdots = 0$
Therefore
$\emptyset(0) = 3, \Delta\, \emptyset(0) = -2, \Delta^2 \emptyset(0) = 0, \Delta^3 \emptyset(0) = 6, \Delta^4 \emptyset(0) = \Delta^5 \emptyset(0) = \cdots = 0$
Now consider $\phi(n)=2n^4+6n^3-3n^2+2$ and interval of differencing in the consecutive values of n is unity then
$\Delta \emptyset(n) = \emptyset(n+1) - \emptyset(n) = 8n^3 + 30n^2 + 20n + 5$
$\Delta^2 \emptyset(n) = 24n^2 + 84n + 58$
$\Delta^3 \emptyset(n) = 48n + 108$
$\Delta^4 \emptyset(n) = 48$
$\Delta^5 \emptyset(n) = \Delta^6 \emptyset(n) = \cdots = 0$
Therefore
$\emptyset(0) = 2, \Delta\, \emptyset(0) = 5, \Delta^2 \emptyset(0) = 58, \Delta^3 \emptyset(0) = 108, \Delta^4 \emptyset(0) = 48, \Delta^5 \emptyset(0) = \Delta^6 \emptyset(0) = \cdots = 0$
Now consider $\phi(n)=n^6+729i$ where $i = \sqrt{-1}$, and interval of differencing in the consecutive values of n is unity then
$\Delta \emptyset(n) = \emptyset(n+1) - \emptyset(n) = 6n^5 + 15n^4 + 20n^3 + 15n^2 + 6n + 1$
$\Delta^2 \emptyset(n) = 30n^4 + 120n^3 + 210n^2 + 180n + 62$
$\Delta^3 \emptyset(n) = 120n^3 + 540n^2 + 900n + 540$
$\Delta^4 \emptyset(n) = 360n^2 + 1440n + 1560$
$\Delta^5 \emptyset(n) = 720n + 1800$
$\Delta^6 \emptyset(n) = 720$
$\Delta^7 \emptyset(n) = \Delta^8 \emptyset(n) = \cdots = 0$
Therefore
$\emptyset(0) = 729i, \Delta\, \emptyset(0) = 1, \Delta^2 \emptyset(0) = 62, \Delta^3 \emptyset(0) = 540, \Delta^4 \emptyset(0) = 1560, \Delta^5 \emptyset(0) = 1800,$
$\Delta^6 \emptyset(0) = 720, \Delta^7 \emptyset(0) = \Delta^8 \emptyset(0) = \cdots = 0$

VI. SOME APPLICATIONS IN HYPERGEOMETRIC REDUCTION FORMULAE

In Eqn. (29), put $\phi(n)=n^4+1$ and in view of the ordinary forward differences, give in the section (V), we get

$$\sum_{n=0}^{\infty} \{(2n)^4 + 1\} \frac{x^{2n}}{(2n)!} = \cosh(x) \left[1 + 14\,\frac{x^2}{2!} + 24\,\frac{x^4}{4!} + 0 + \cdots\right] + \sinh(x) \left[\frac{x}{1!} + 36\,\frac{x^3}{3!} + 0 + \cdots\right]$$
$$= \cosh(x)\ (1 + 7x^2 + x^4) + \sinh(x)\,(x + 6x^3)$$

which is the well known result of S. Narayana Aiyar [1].
In veiw of the section (IV) for factorization, we get

$$(2n)^4 + 1 = \left[2n + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]\left[2n + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right]\left[2n - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]\left[2n - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right]$$

Now applying the Lemma (7), we get

$$\sum_{n=0}^{\infty} \frac{\left(\frac{2\sqrt{2}-1-i}{2\sqrt{2}}\right)_n \left(\frac{2\sqrt{2}-1+i}{2\sqrt{2}}\right)_n \left(\frac{2\sqrt{2}+1-i}{2\sqrt{2}}\right)_n \left(\frac{2\sqrt{2}+1+i}{2\sqrt{2}}\right)_n}{\left(\frac{1}{2}\right)_n \left(\frac{-1-i}{2\sqrt{2}}\right)_n \left(\frac{-1+i}{2\sqrt{2}}\right)_n \left(\frac{1-i}{2\sqrt{2}}\right)_n \left(\frac{1+i}{2\sqrt{2}}\right)_n} \frac{\left(\frac{x^2}{4}\right)^n}{(n)!} = (1+7x^2+x^4)\cosh(x) + (x+6x^3)\sinh(x)$$

Now representing the above power series in hypergeometric forms we get equation (8).

Similarly if we put $\phi(n)=n^4+1$ in equation (30) and in view of the sections (IV) and (V), and proceeding on same parallel lines for derivation of equation (8), we get Eqn. (9).

In Eqns. (29) and (30), put $\phi(n)=n^5+1$ and in view of the relations given in the sections (IV) and (V), we get Eqns. (10) and (11) respectively.

In Eqns. (29) and (30), put $\phi(n)=3n^3-4n^2+n+88$ and in view of sections (IV) and (V), we get Eqns. (12) and (13) respectively.

In Eqns. (29) and (30), put $\phi(n)=n^3-3n^2+3$ and in view of sections (IV) and (V), we get Eqns. (14) and (15) respectively.

In Eqns. (29) and (30), put $\phi(n)=2n^4+6n^3-3n^2+2$ and in view of sections (IV) and (V), we get Eqns. (16) and (17) respectively.

In Eqns. (29) and (30), put $\phi(n)=n^6+729i$ and in view of sections (IV) and (V), we get Eqns. (18) and (19) respectively.

VII. FUTURE SCOPE

As we see these summation theorem are very useful for the summation of some infinite series. We can further generate summation theorem for some parameters like ${}_7F_8$, ${}_8F_9$, ${}_9F_{10}$ etc. whose numerator and denominator parameter are complex.

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